EE120 - Fall'19 - Lecture 17 Notes[1]

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Simple Bode Plots

Bode plots are straight-line approximations to the magnitude and phase of the frequency response, laid on a logarithmic scale. They are useful for building intuition about what type of frequency response to expect from a given pole/zero configuration.

Example 1: As a first example consider the transfer function

$$H(s) = \frac{1}{\tau s + 1}$$

which has a single pole at $s = -1/\tau$ and DC gain $H(0) = 1$. Note

$$H(j\omega) = \frac{1}{j\omega\tau + 1} \quad \Rightarrow \quad |H(j\omega)| = \frac{1}{\sqrt{(\omega\tau)^2 + 1}}.$$

Since $(\omega\tau)^2 + 1 \approx (\omega\tau)^2$ when $\omega\tau \gg 1$ and ≈ 1 when $\omega\tau \ll 1$,

$$|H(j\omega)| \approx \begin{cases} \frac{1}{\omega\tau} & \omega\tau \gg 1 \\ 1 & \omega\tau \ll 1. \end{cases}$$

The decibel (dB) scale converts $|H(j\omega)|$ to its logarithmic value (base 10) and multiplies it by 20. Thus,

$$20\log_{10}|H(j\omega)| \approx \begin{cases} -20\log_{10}\omega - 20\log_{10}\tau & \omega\tau \gg 1 \\ 0 \text{ dB} & \omega\tau \ll 1. \end{cases}$$

This approximation is depicted in Figure 1 (top) as a function of $\log_{10}\omega$. Note that the largest approximation error occurs when $\omega = 1/\tau$, where $|H(j\omega)| = 1/\sqrt{2}$, thus $20\log_{10}|H(j\omega)| \approx -3$ dB.

The phase plot is shown at the bottom of Figure 1. It approximates $\measuredangle H(j\omega)$ with $0°$ when $\omega\tau \leq 0.1$, with $-90°$ when $\omega\tau \geq 10$, and with a straight line connecting $0°$ to $-90°$ in between. This approximation matches the exact value $-45°$ when $\omega\tau = 1$.

Example 2: Now let

$$H(s) = \tau s + 1$$

which is the reciprocal of the transfer function in Example 1. Thus, the phase $\measuredangle H(j\omega)$ has the opposite sign of the phase in Example 1. Likewise, since $|H(j\omega)|$ is the reciprocal of the magnitude in Example 1, $20\log_{10}|H(j\omega)|$ has the opposite sign. The resulting Bode plots for the magnitude and phase are shown in Figure 2.

Example 3: For $H(s) = K > 0$, a constant gain, the Bode plots are as shown in Figure 3. For $K < 0$ the phase must be replaced with $180°$.

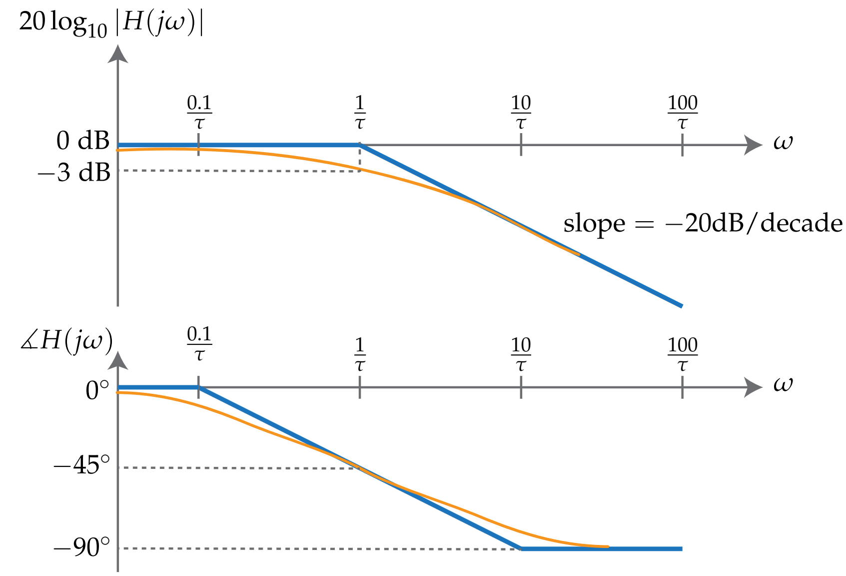


Figure 1: Bode magnitude and phase plots for

$$H(s) = \frac{1}{\tau s + 1}.$$

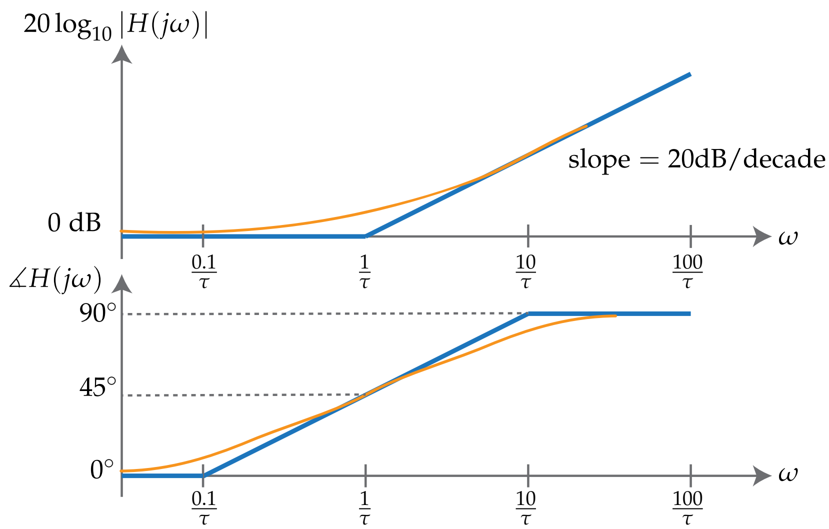


Figure 2: Bode magnitude and phase plots for

$$H(s) = \tau s + 1.$$

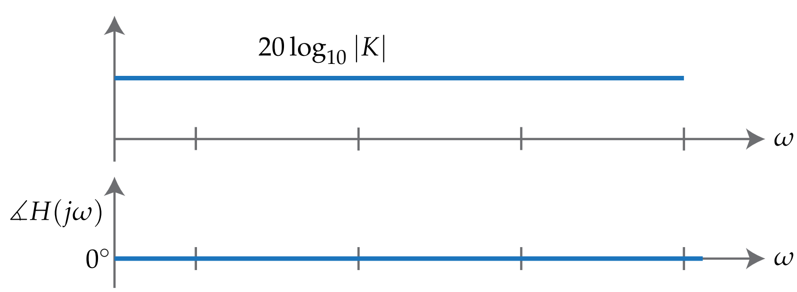


Figure 3: Bode magnitude and phase plots for

$$H(s) = K > 0.$$

For $K < 0$ the phase must be replaced with $180°$.

Bode Plots for Transfer Functions with Multiple Poles and Zeros

Above we saw Bode plots for a single pole, a single zero, and a constant gain. We now consider a transfer function with N poles, M zeros, and a DC gain K that is not necessarily 1:

$$H(s) = K\frac{(\tau_1 s+1)\cdots(\tau_M s+1)}{(\tau_{M+1}s+1)\cdots(\tau_{M+N}s+1)}. \tag{1}$$

Since

$$|H(j\omega)| = |K|\frac{|j\omega\tau_1+1|\cdots|j\omega\tau_M+1|}{|j\omega\tau_{M+1}+1|\cdots|j\omega\tau_{M+N}+1|},$$

we have:

$$\begin{aligned} 20\log_{10}|H(j\omega)| &= 20\log_{10}|K| + \sum_{i=1}^{M} 20\log_{10}|j\omega\tau_i+1| \\ &\quad - \sum_{i=M+1}^{M+N} 20\log_{10}|j\omega\tau_i+1|. \end{aligned}$$

This means that we can construct the Bode magnitude plot for (1) by adding the plots in Examples 1-3 for single poles, zeros, and gains. Likewise, when $K > 0$, the phase of (1) is

$$\measuredangle H(j\omega) = \sum_{i=1}^{M}\measuredangle(j\omega\tau_i+1) - \sum_{i=M+1}^{M+N}\measuredangle(j\omega\tau_i+1),$$

which means we can add the phase plots for single poles and zeros. When $K < 0$ we add $180°$ to the expression above to account for the negative sign.

Example 4: To draw the Bode plots for

$$H(s) = \frac{s+1}{s+10}$$

we first bring it to the form (1):

$$H(s) = 0.1\frac{s+1}{0.1s+1}$$

where $K = 0.1$ is the DC gain.

Figure 4 (top) shows the Bode magnitude plot (solid lines), obtained by adding the dashed plots for the single pole, zero, and gain $20\log_{10} K = -20$ dB. The bottom plot shows the phase (solid lines), obtained by adding the dashed phase plots for the pole and zero.

Compare these Bode plots to the frequency response plots we obtained for the same transfer function in Lecture 16, using a geometric interpretation of the frequency response.

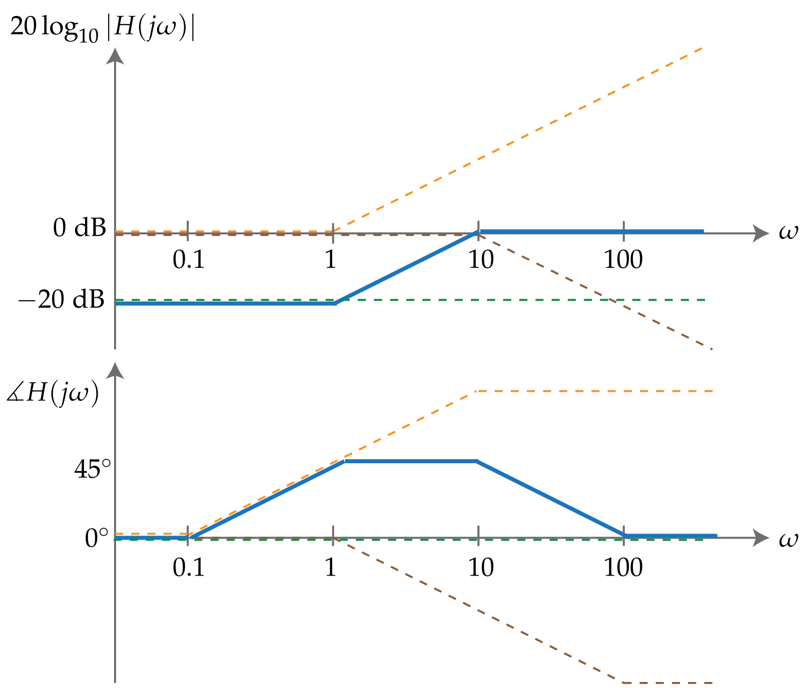


Figure 4: Bode magnitude and phase plots (solid lines) for

$$H(s) = \frac{s+1}{s+10},$$

obtained by adding the dashed lines, which represent the corresponding plots for the zero (orange), pole (brown) and the gain $K = 0.1$ (green).

Example 5: Consider

$$H(s) = \frac{1}{(\tau s + 1)^2} \tag{2}$$

which has two identical poles and DC gain = 1. Thus, the Bode plots are obtained by doubling the magnitude and phase the plots in Figure 1 for a single pole. The resulting Bode plots are shown in Figure 5.

Example 6: (Complex poles) Now let

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{3}$$

which has complex conjugate poles at $s = \omega_n(-\zeta \pm j\sqrt{1-\zeta^2})$ for $\zeta \in [0, 1)$ and two real poles at $s = -\omega_n$ when $\zeta = 1$. Indeed, for $\zeta = 1$ the denominator is the perfect square $(s + \omega_n)^2$, and (3) can be rewritten as (2) with

$$\tau = \frac{1}{\omega_n}.$$

The Bode plot for (3) is drawn with the simplifying assumption that $\zeta = 1$; that is, it has the same form as Figure 5 drawn for Example 5. This is a reasonable approximation when ζ is close to 1. For smaller ζ the approximation agrees with the true frequency response asymptotically as $\omega \to \infty$ and $\omega \to 0$, but it deteriorates around the frequency $\omega = \omega_n$. For example it fails to capture resonance peaks that occur when $\zeta < 0.7$ (compare Figure 5 below to Figure 3 in Lecture 16).

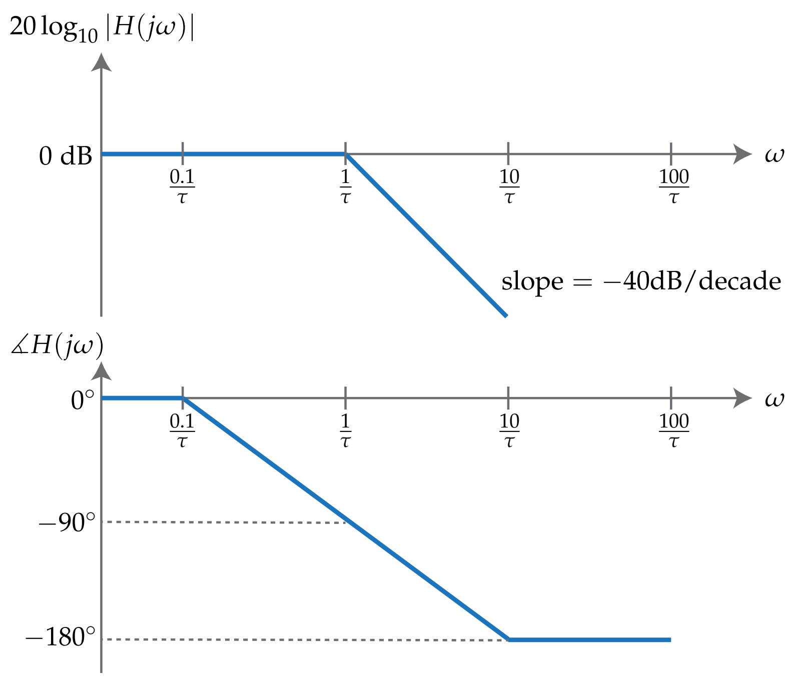
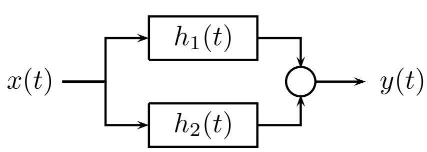


Figure 5: Bode magnitude and phase plots for

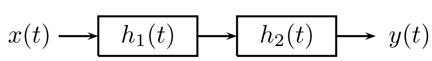
$$H(s) = \frac{1}{(\tau s+1)^2}.$$

Transfer Functions of Interconnected LTI Systems



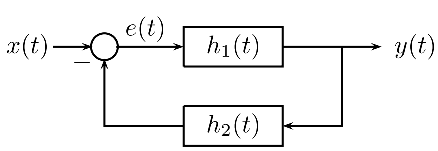
$$h(t) = h_1(t) + h_2(t)$$

$$\boxed{H(s) = H_1(s) + H_2(s)}$$



$$h(t) = (h_1 * h_2)(t)$$

$$\boxed{H(s) = H_1(s)H_2(s)}$$

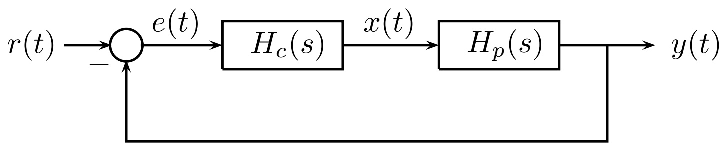


$$\begin{aligned} E(s) &= X(s) - H_2(s)Y(s) \\ Y(s) &= H_1(s)E(s) \\ &= H_1(s)X(s) - H_1(s)H_2(s)Y(s) \end{aligned}$$

$$(1 + H_1(s)H_2(s))Y(s) = H_1(s)X(s)$$

$$\boxed{\frac{Y(s)}{X(s)} = H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}}$$

Example: Feedback Control



$r(t)$: reference signal to be tracked by $y(t)$
$H_c(s)$: controller, $H_p(s)$: system to be controlled - "plant"

$$H(s) = \frac{H_c(s)H_p(s)}{1 + H_c(s)H_p(s)}$$

Transfer Functions from State Space Models

As you saw in EECS 16B, state space representations are commonly used for constant coefficient linear differential equations. In this representation the "state vector" $\vec{z}(t) \in \mathbb{R}^n$ evolves according to the vector differential equation

$$\frac{d}{dt}\vec{z}(t) = A\vec{z}(t) + Bx(t) \tag{4}$$

where $x(t) \in \mathbb{R}$ is the input, and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$. The output $y(t) \in \mathbb{R}$ is a linear function of $\vec{z}(t)$ and $x(t)$:

$$y(t) = C\vec{z}(t) + Dx(t), \tag{5}$$

with $C \in \mathbb{R}^{1\times n}$, $D \in \mathbb{R}$.

To obtain the transfer function from this representation we apply the Laplace Transform to both sides of (4)-(5) and obtain:

$$\begin{aligned} s\vec{Z}(s) &= A\vec{Z}(s) + BX(s) & (6) \\ Y(s) &= C\vec{Z}(s) + DX(s). & (7) \end{aligned}$$

We rearrange (6) as

$$(sI - A)\vec{Z}(s) = BX(s),$$

where the scalar s is multiplied with the $n \times n$ identity matrix to match the dimension of A, and obtain

$$\vec{Z}(s) = (sI - A)^{-1}BX(s).$$

Substituting in (7) we get

$$Y(s) = C(sI - A)^{-1}BX(s) + DX(s) = [C(sI - A)^{-1}B + D]X(s),$$

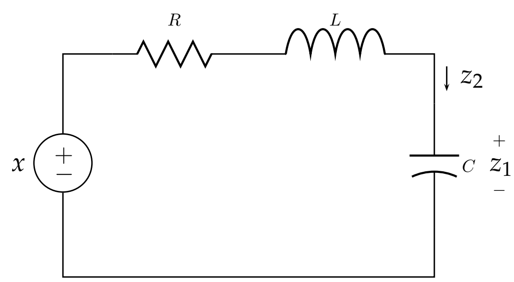
which means that the transfer function is

$$\boxed{H(s) = C(sI - A)^{-1}B + D.}$$

Note that $(sI - A)^{-1}$ is well-defined when $sI - A$ is full-rank, thus invertible. Since the values of s for which $sI - A$ drops rank are the eigenvalues of A, $H(s)$ has a singularity at each eigenvalue. Thus:

$$\boxed{\text{The poles of } H(s) \text{ are the eigenvalues of } A.}$$

Example:



Consider the RLC circuit above and let $z_1(t)$ be the voltage across the capacitor and and let $z_2(t)$ be the current. Then $C\frac{dz_1(t)}{dt} = z_2(t)$ and $L\frac{dz_2(t)}{dt}$ is the voltage across the inductor, which is equal to $x(t) - z_1(t) - Rz_2(t)$ by Kirchhoff's Voltage Law. Thus, the state equations are:

$$\frac{dz_1(t)}{dt} = \frac{1}{C}z_2(t) \tag{8}$$

$$\frac{dz_2(t)}{dt} = -\frac{1}{L}z_1(t) - \frac{R}{L}z_2(t) + \frac{1}{L}x(t) \tag{9}$$

which is of the form (4) with

$$A = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}.$$

If we define the output to be the current $y = z_2$, then

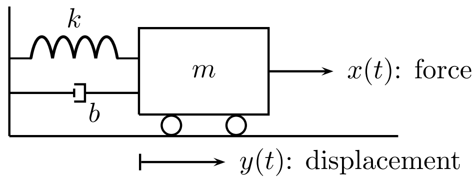
$$\overline{C} = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad \text{and} \quad D = 0$$

where we used the notation $\overline{C}$ for the row vector in (5) to distinguish it from the capacitance C. Then,

$$H(s) = \overline{C}(sI - A)^{-1}B + D = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s & -\frac{1}{C} \\ \frac{1}{L} & s + \frac{R}{L} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} = \frac{\frac{1}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}.$$

Note that the denominator of $H(s)$ is the characteristic polynomial of A, which confirms that the poles of $H(s)$ are the eigenvalues of A.

Example: Consider the mass-damper-spring system below.



Let $z_1(t) = y(t)$ denote the displacement and $z_2(t) = \frac{dy(t)}{dt}$ the velocity. Then the state equations are:

$$\frac{dz_1(t)}{dt} = z_2(t) \tag{10}$$

$$\frac{dz_2(t)}{dt} = \frac{1}{m}\left(-kz_1(t) - bz_2(t) + x(t)\right) \tag{11}$$

where the second equation is obtained by matching the net force to mass times the acceleration. The state equations (10)-(11) can be brought to the form (4) with

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}.$$

Since $y = z_1$, we have

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

and

$$H(s) = C(sI - A)^{-1}B = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{b}{m} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} = \frac{\frac{1}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}}.$$